Network formation models

I. Makarov & L.E. Zhukov

BigData Academy MADE from Mail.ru Group

Social Network Analysis and Machine Learning on Graphs

Network models

Empirical network features:

- Power-law (heavy-tailed) degree distribution
- Small average distance (graph diameter)
- Large clustering coefficient (transitivity)

Generative models:

- Random graph model (Erdos & Renyi, 1959)
- Preferential attachment model (Barabasi & Albert, 1999)
- "Small world" model (Watts & Strogatz, 1998)

Random graph model

Graph $G\{E, V\}$, nodes $n = |V|$, edges $m = |E|$
Erdos and Renyi, 1959.
Random graph models

- $G_{n,m}$, a randomly selected graph from the set of $C_N^m graphs, N = \frac{n(n-1)}{2}$, with n nodes and m edges
- $G_{n,p}$, each pair out of $N = \frac{n(n-1)}{2}$ pairs of nodes is connected with probability p, m - random number

$$\langle m \rangle = p\frac{n(n-1)}{2}$$

$$\langle k \rangle = \frac{1}{n}\sum_i k_i = \frac{2\langle m \rangle}{n} = p\ (n-1) \approx pn$$

$$\rho = \frac{\langle m \rangle}{n(n-1)/2} = p$$

Random graph model

- Probability that i-th node has a degree $k_i = k$

$$P(k_i = k) = P(k) = C_{n-1}^k p^k (1-p)^{n-1-k}$$

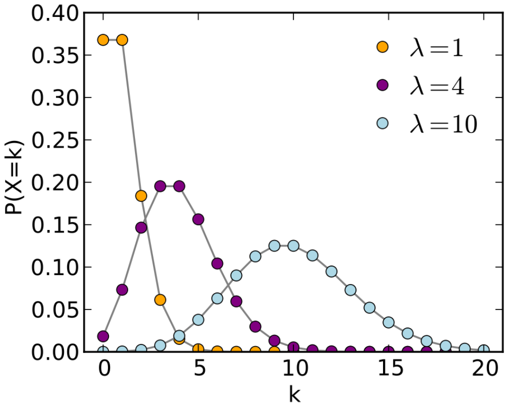
  (Bernoulli distribution)
  p^k - probability that connects to k nodes (has k-edges)
  $(1-p)^{n-k-1}$ - probability that does not connect to any other node
  C_{n-1}^k - number of ways to select k nodes out of all to connect to

- Limiting case of Bernoulli distribution, when $n \to \infty$ at fixed $\langle k \rangle = pn = \lambda$

$$P(k) = \frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!} = \frac{\lambda^k e^{-\lambda}}{k!}$$

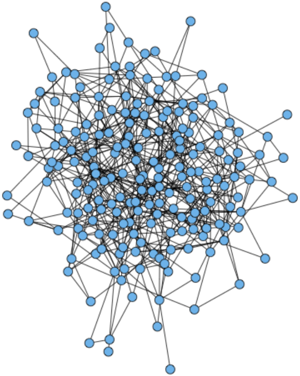
  (Poisson distribution)

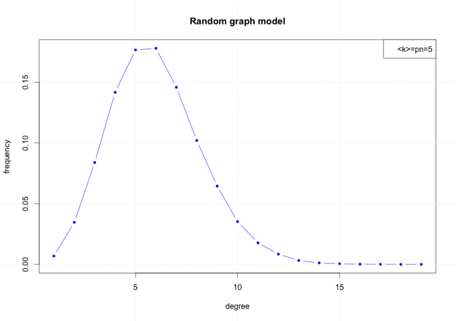
Poisson Distribution



$$P(k_i = k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad \lambda = pn$$

Random graph



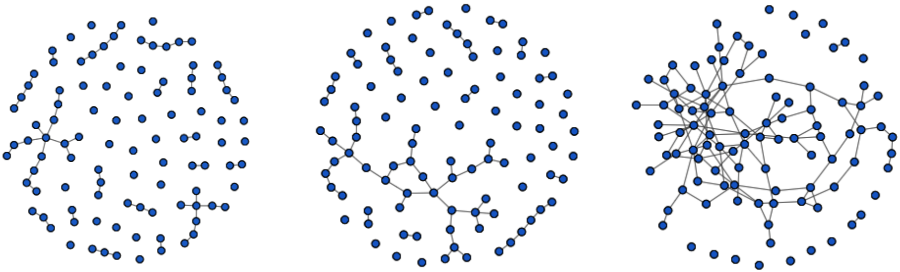


$$\langle k \rangle = pn = 5$$

Random graph model

Consider $G_{n,p}$ as a function of p

- $p = 0$, empty graph - $\langle k \rangle = 0$
- $p = 1$, complete (full) graph - $\langle k \rangle = n - 1$
- n_G -largest connected component, $s = \frac{n_G}{n}$



p

Phase transition

Let u – fraction of nodes that do not belong to GCC. The probability that a node does not belong to GCC

$$u = \frac{n - n_G}{n} = P(k=0) + P(k=1) \cdot u + P(k=2) \cdot u^2 + P(k=3) \cdot u^3 ... =$$

$$= \sum_{k=0}^{\infty} P(k) u^k = \sum_{k=0} \frac{\lambda^k e^{-\lambda}}{k!} u^k = e^{-\lambda} e^{\lambda u} = e^{\lambda(u-1)}$$

Let s -fraction of nodes belonging to GCC (size of GCC)

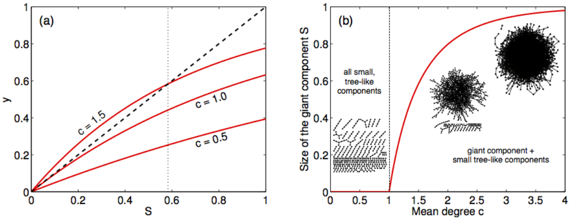
$$s = 1 - u$$

$$1 - s = e^{-\lambda s}$$

when $\lambda \to \infty, \quad s \to 1$
when $\lambda \to 0, \quad s \to 0$
$\lambda = pn = \langle k \rangle$

Phase transition

$$s = 1 - e^{-\lambda s}$$



non-zero solution exists when (at $s = 0$):

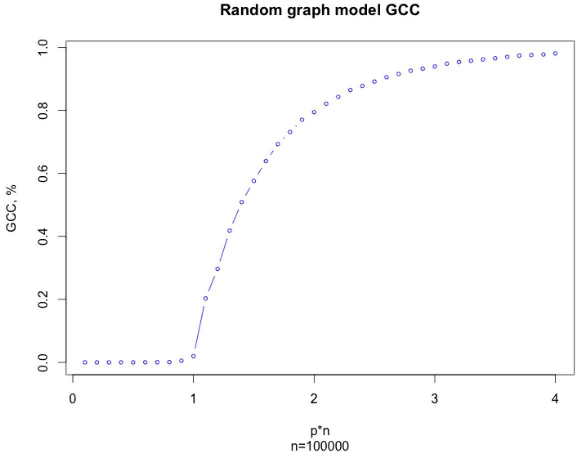
$$\lambda e^{-\lambda s} > 1$$

critical value:

$$\lambda_c = 1$$

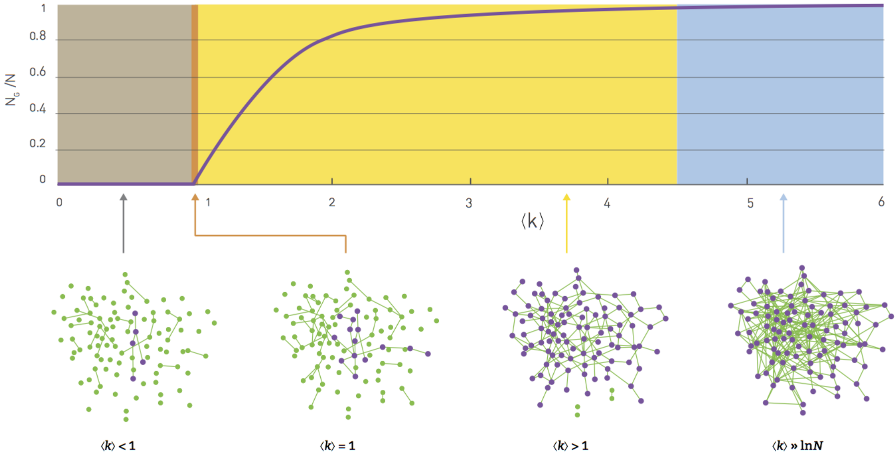
$$\lambda_c = \langle k \rangle = p_c n = 1, \quad p_c = \frac{1}{n}$$

Numerical simulations



$$\langle k \rangle = pn$$

evolution of random network



from A-L. Barabasi, 2016

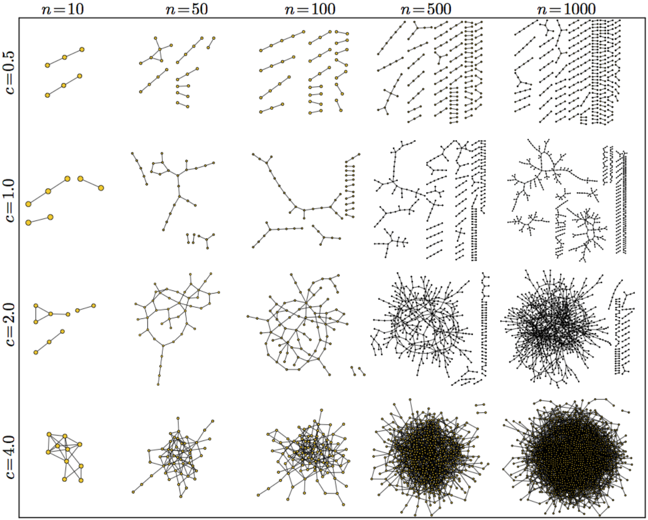
Phase transition

Graph $G(n, p)$, for $n \to \infty$, critical value $p_c = 1/n$

- Subcritical regime: $p < p_c$, $\langle k \rangle < 1$ there is no components with more than $O(\ln n)$ nodes, largest component is a tree
- Critical point: $p = p_c$, $\langle k \rangle = 1$ the largest component has $O(n^{2/3})$ nodes
- Supercritical regime: $p > p_c$, $\langle k \rangle > 1$ gigantic component has all $O((p - p_c)n)$ nodes
- Connected regime: $p >> \ln n/n$, $\langle k \rangle > \ln n$ gigantic component has all $O(n)$ nodes

Critical value: $\langle k \rangle = p_c n = 1$- on average one neighbor for a node

Numerical simulation

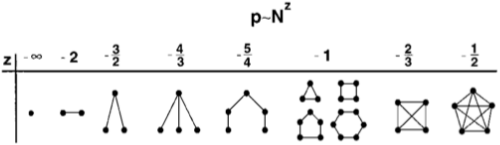


Clauset, 2014

Threshold probabilities

Graph $G(n, p)$

Threshold probabilities when different subgraphs of k-nodes and l-edges appear in a random graph $p_s \sim n^{-k/l}$



When $p > p_s$:

- $p_s \sim n^{-k/(k-1)}$, having a tree with k nodes
- $p_s \sim n^{-1}$, having a cycle with k nodes
- $p_s \sim n^{-2/(k-1)}$, complete subgraph with k nodes

Barabasi, 2002

Clustering coefficient

- Clustering coefficient (probability that two neighbors link to each other):

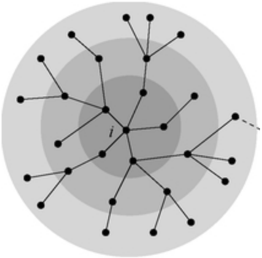
$$C_i(k) = \frac{\#\text{of links between NN}}{\#\text{max number of links NN}} = \frac{pk(k-1)/2}{k(k-1)/2} = p$$

$$C = p = \frac{\langle k \rangle}{n}$$

- when $n \to \infty, \quad C \to 0$

Graph diameter

- $G(n, p)$ is locally tree-like (GCC) (no loops; low clustering coefficient)



- on average, the number of nodes d steps away from a node

$$n = 1 + \langle k \rangle + \langle k \rangle^2 + ... \langle k \rangle^D = \frac{\langle k \rangle^{D+1} - 1}{\langle k \rangle - 1} \approx \langle k \rangle^D$$

- in GCC, around p_c , $\langle k \rangle^D \sim n$,

$$D \sim \frac{\ln n}{\ln \langle k \rangle}$$

Random graph model

- Node degree distribution function - Poisson:

$$P(k) = \frac{\lambda^k e^{-\lambda}}{k!}, \quad \lambda = pn = \langle k \rangle$$

- Average path length:

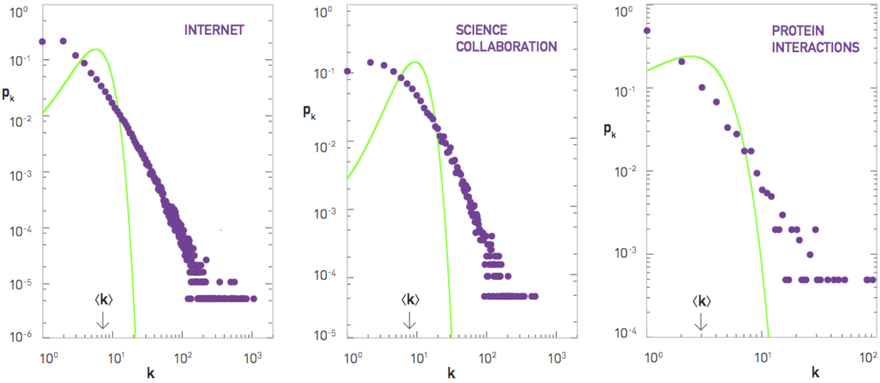
$$\langle L \rangle = \frac{\ln n}{\ln \langle k \rangle}$$

- Clustering coefficient:

$$C = \frac{\langle k \rangle}{n}$$

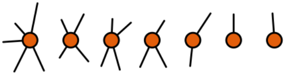
Real networks

Degree distribution in real networks



Configuration model

- Random graph with n nodes with a given degree sequence: $D = \{k_1, k_2, k_3..k_n\}$ and $m = 1/2 \sum_i k_i$ edges.
- Construct by randomly matching two stubs and connecting them by an edge.



- Can contain self loops and multiple edges
- Probability that two nodes i and j are connected

$$p_{ij} = \frac{k_i k_j}{2m - 1}$$

- Will be a simple graph for special "graphical degree sequence"

Motivation of Evolutional Random Graph Model

Most of the networks we study are evolving over time, they expand by adding new nodes:

- Citation networks
- Collaboration networks
- Web
- Social networks

Preferential attachment model

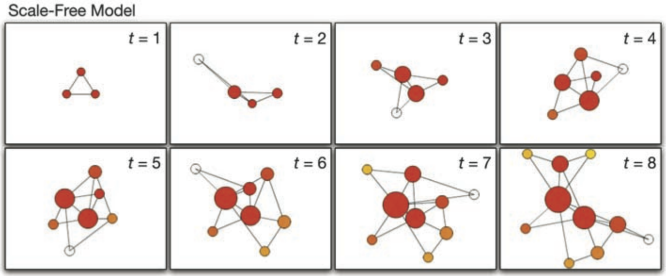
Barabasi and Albert, 1999
Dynamic growth model
Start at $t = 0$ with n_0 nodes and some edges $m_0 \geq n_0$

1. **Growth**
   At each time step add a new node with m edges ($m \leq n_0$), connecting to m nodes already in netwrok $k_i(i) = m$
2. **Preferential attachment**
   The probability of linking to existing node i is proportional to the node degree k_i

$$\Pi(k_i) = \frac{k_i}{\sum_i k_i}$$

after t timesteps: $t + n_0$ nodes, $mt + m_0$ edges

Preferential attachment model



Barabasi, 1999

Preferential attachment

Continues approximation: continues time, real variable node degree
$\langle k_i(t) \rangle$- expected value over multiple realizations
Time-dependent degree of a single node:

$$k_i(t + \delta t) = k_i(t) + m\Pi(k_i)\delta t$$

$$\frac{dk_i(t)}{dt} = m\Pi(k_i) = m\frac{k_i}{\sum_i k_i} = \frac{mk_i}{2mt}$$

$$\frac{dk_i(t)}{dt} = \frac{k_i(t)}{2t}$$

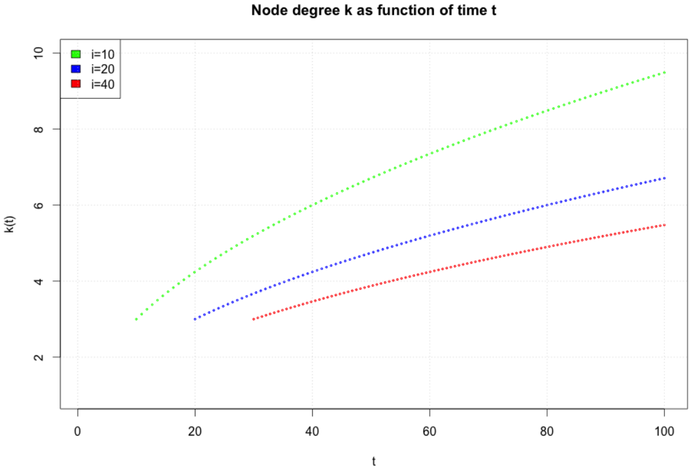
initial conditions: $k_i(t = i) = m$

$$\int_m^{k_i(t)} \frac{dk_i}{k_i} = \int_i^t \frac{dt}{2t}$$

Solution:

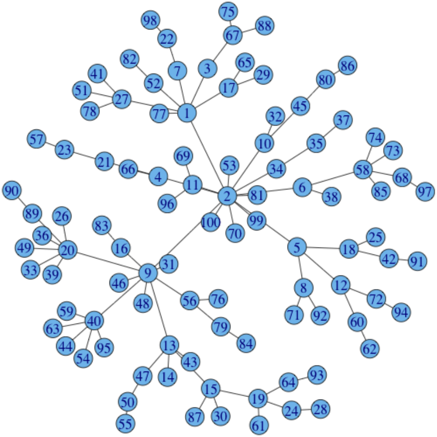
$$k_i(t) = m\left(\frac{t}{i}\right)^{1/2}$$

Preferential attachement



$$k_i(t) = m\left(\frac{t}{i}\right)^{1/2}$$

Preferential attachement



Preferential attachment

Time evolution of a node degree

$$k_i(t) = m \left(\frac{t}{i}\right)^{1/2}$$

Nodes with $k_i(t) \leq k$:

$$m \left(\frac{t}{i}\right)^{1/2} \leq k$$

$$i \geq \frac{m^2}{k^2} t$$

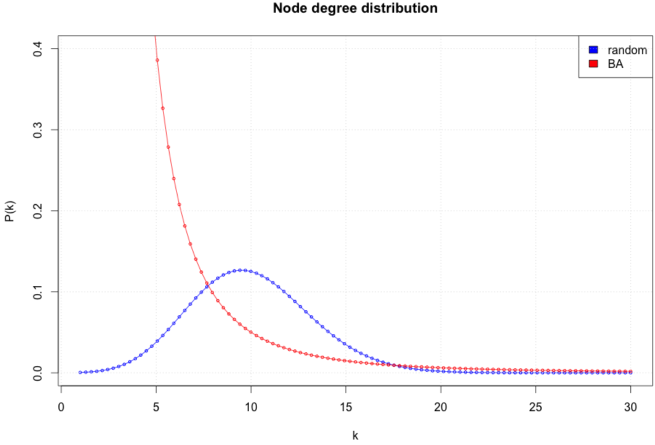
Probability of randomly selected node to have $k' \leq k$ (fraction of nodes with $k' \leq k$)

$$F(k) = P(k' \leq k) = \frac{n_0 + t - m^2 t / k^2}{n_0 + t} \approx 1 - \frac{m^2}{k^2}$$

Distribution function:

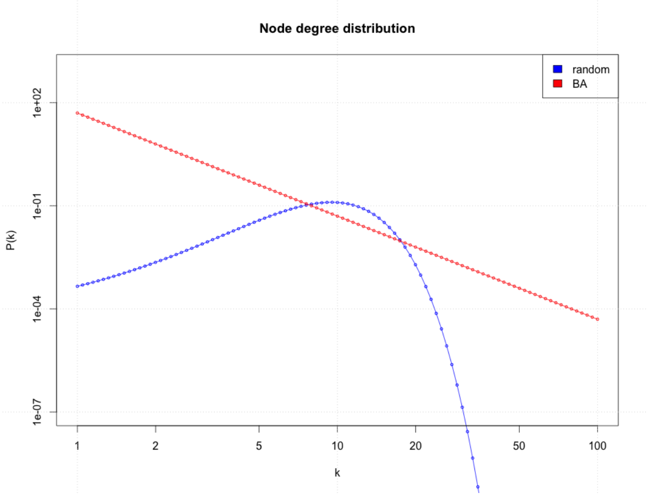
$$P(k) = \frac{d}{dk} F(k) = \frac{2m^2}{k^3}$$

Preferential attachment vs random graph



$$BA : P(k) = \frac{2m^2}{k^3}, \qquad ER : P(k) = \frac{\langle k \rangle^k e^{-\langle k \rangle}}{k!}, \quad \langle k \rangle = pn$$

Preferential attachment vs random graph



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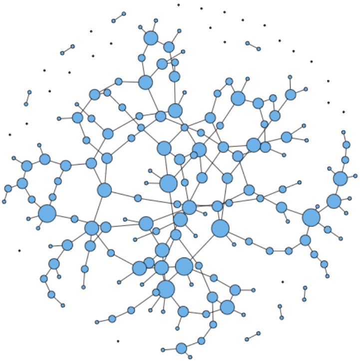
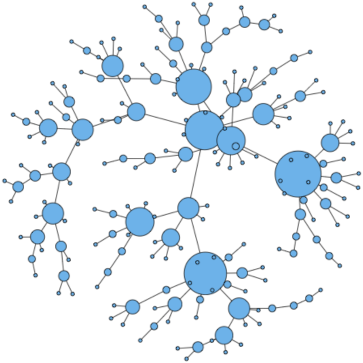
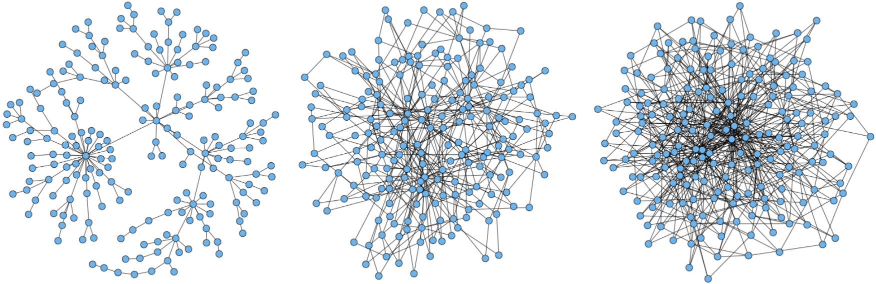
Preferential attachment vs random graph





Preferential attachment model



$m = 1$ $m = 2$ $m = 3$

Growing random graph

1. **Growth**
   At each time step add a new node with m edges ($m \leq n_0$), connecting to m nodes already in network $k_i(i) = m$
2. **~~Preferential attachment~~ Uniformly at random**
   The probability of linking to existing node i is

$$\Pi(k_i) = \frac{1}{n_0 + t - 1}$$

Node degree growth:

$$k_i(t) = m\left(1 + \log\left(\frac{t}{i}\right)\right)$$

Node degree distribution function:

$$P(k) = \frac{e}{m} \exp\left(-\frac{k}{m}\right)$$

Preferential attachment

- Power law distribution function:

$$P(k) = \frac{2m^2}{k^3}$$

- Average path length (analytical result) :

$$\langle L \rangle \sim \log(N)/\log(\log(N))$$

- Clustering coefficient (numerical result):

$$C \sim N^{-0.75}$$

Many more models

Some other models that produce scale-free networks:

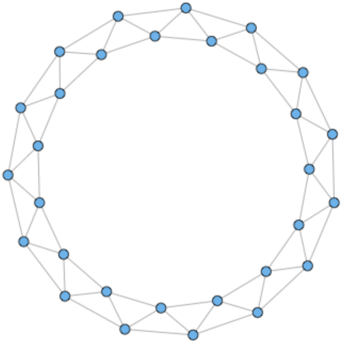
- Non-linear preferential attachment
- Link selection model
- Copying model
- Cost-optimization model
- ...

Historical note

- Polya urn model, George Polya, 1923
- Yule process, Udny Yule, 1925
- Distribution of wealth, Herbert Simon,1955
- Evolution of citation networks, cumulative advantage, Derek de Solla Price, 1976
- Preferential attachment network model, Barabasi and Albert, 1999

Small world

Motivation: keep high clustering, get small diameter



Clustering coefficient $C = 1/2$
Graph diameter $d = 8$

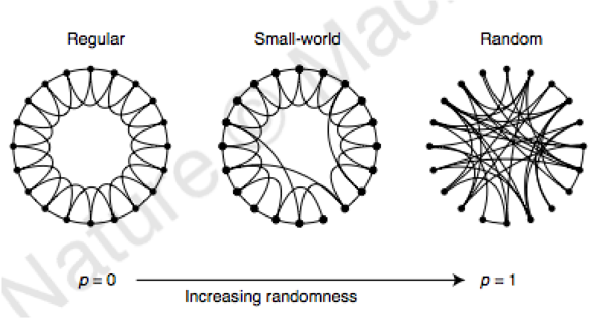
Small world

Watts and Strogatz, 1998

Single parameter model, interpolation between regular lattice and random graph

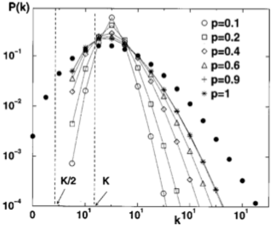
- start with regular lattice with n nodes, k edges per vertex (node degree), $k << n$
- randomly connect with other nodes with probability p, forms $pnk/2$ "long distance" connections from total of $nk/2$ edges
- $p = 0$ regular lattice, $p = 1$ random graph

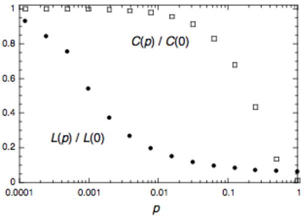
Small world



Watts, 1998

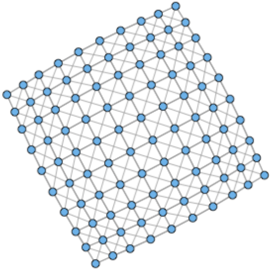
- Node degree distribution:
  Poisson like
- Ave. path length $\langle L(p) \rangle$:
  $p \to 0$, ring lattice, $\langle L(0) \rangle = 2n/k$
  $p \to 1$, random graph, $\langle L(1) \rangle = \log(n)/\log(k)$
- Clustering coefficient $C(p)$:
  $p \to 0$, ring lattice, $C(0) = 3/4 = const$
  $p \to 1$, random graph, $C(1) = k/n$

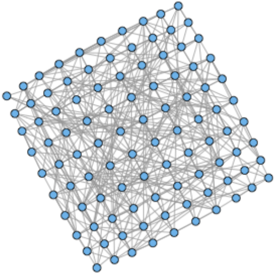




Watts, 1998

Small world model





20% rewiring:

ave. path length = 3.58 $\rightarrow$ ave. path length = 2.32

clust. coeff = 0.49 $\rightarrow$ clust. coeff = 0.19

Model comparison

	Random	BA model	WS model	Empirical networks
P(k)	$\frac{\lambda^k e^{-\lambda}}{k!}$	k^{-3}	poisson like	power law
C	$\langle k \rangle / N$	$N^{-0.75}$	const	large
$\langle L \rangle$	$\frac{\log(N)}{\log(\langle k \rangle)}$	$\frac{\log(N)}{\log \log(N)}$	$\log(N)$	small

References

- On random graphs I, P. Erdos and A. Renyi, Publicationes Mathematicae 6, 290–297 (1959).
- On the evolution of random graphs, P. Erdos and A. Renyi, Publicaton of the Mathematical Institute of the Hungarian Academy of Sciences, 17-61 (1960)
- Emergence of Scaling in Random Networks, A.L. Barabasi and R. Albert, Science 286, 509-512, 1999
- Collective dynamics of small-world networks. Duncan J. Watts and Steven H. Strogatz. Nature 393 (6684): 440-442, 1998